3 Orbital Mechanics

Isaac Newton (1642/3–1727) was born in rural England; his birth date was 1642 December 25 according to the Julian calendar (still in use in England at the time), but 1643 January 4 according to the Gregorian calendar. When young Newton proved to be incompetent at managing his family's farm, he was sent to Cambridge University and started to thrive as a scholar. In 1665, the year in which Newton earned his bachelor's degree, an outbreak of the plague closed down the university, and Newton retreated to his family's farm and began to think—very hard. The period when the university was closed was Newton's *annus mirabilis*, during which he discovered calculus, formulated his three **laws of motion** and his **law of universal gravitation**, and performed ground-breaking experiments in optics. Much of the remainder of Newton's long life was dedicated to developing the ideas he had in this burst of youthful creativity.[1]

Newton didn't publish his laws of motion and law of universal gravitation until 1687, when his book *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy") was published. The laws of motion can be summarized as follows:

1. An object's velocity remains constant unless a net outside force acts upon it.
2. If a net outside force acts on an object, its acceleration is directly proportional to the force and inversely proportional to the mass of the object. In short, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ is the outside force, m is the mass, and $\vec{\mathbf{a}}$ is the acceleration.
3. Forces come in pairs, equal in magnitude and opposite in direction. (As Newton put it: *Actioni contrariam semper et aequalem esse reactionem*, or "Every action has an equal and opposite reaction.")

Newton's law of universal gravitation can be concisely expressed in mathematical form. Suppose that two spherical objects, of mass M and m, are separated by a distance r.

[1] He also performed many alchemical experiments while trying to systematize chemistry in the way he did physics, not to mention writing reams of theological works, becoming Master of the Royal Mint, and serving as president of the Royal Society for nearly a quarter-century.

(The distance r is measured between the centers of the two objects.) Newton's law tells us that the gravitational attraction between the two objects is

$$F = -\frac{GMm}{r^2}, \tag{3.1}$$

where G, called the **gravitational constant**, is a universal constant whose value is $G = 6.67 \times 10^{-11}\,\mathrm{Nm^2\,kg^{-2}}$ (where N stands for newton).[2] The negative sign in equation (3.1) tells us that gravity is always an attractive force.

3.1 ▪ DERIVING KEPLER'S LAWS

Newton derived the form of equation (3.1) by requiring that the force of gravity result in planetary orbits that obey Kepler's laws of planetary motion. Newton was solving the problem in the difficult direction: he deduced the form of the law of gravitation starting from the observations. Since we aren't as smart as Newton, we will take the easier direction in the following section; starting with Newton's law of universal gravitation, we'll show that Kepler's laws follow as a consequence. Although it may seem numerically incongruous, the derivations will flow more smoothly if we begin by deriving Kepler's second law, then go on to the first and third laws.

3.1.1 Kepler's Second Law

Gravity is an example of a **central force**, defined as a force directed straight toward or away from some central point, with a magnitude that depends only on the distance r from that point. The gravitational force qualifies as a central force because the force $\vec{\mathbf{F}}$ acting on the mass m always points toward the mass M (the central point of the force), and the magnitude of the gravitational force is $\propto 1/r^2$, where r is the separation of the two masses.[3] While analyzing the motion of a particle responding to a central force, it is convenient to be able to switch from Cartesian coordinates to polar coordinates.

In a Cartesian coordinate system (Figure 3.1), the unit vectors along the x, y, and z axes are $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, and $\hat{\mathbf{k}}$, respectively. Suppose we choose our Cartesian coordinate axes such that the larger mass M lies at the origin, and the position $\vec{\mathbf{r}}$ and velocity $\vec{\mathbf{v}}$ of the smaller mass m lie in the xy plane. (For the sake of concreteness, let's call mass M the Sun, and mass m a planet, although the situation applies in general to any system of two spherical masses: a planet and a moon, a planet and an artificial satellite, a supermassive black hole and a star—you name it.) The planet's position (x, y) can now be expressed in polar coordinates, where the polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) by the relations $x = r\cos\theta$ and $y = r\sin\theta$. In polar coordinates, as illustrated in

[2] The newton (N)—the force required to accelerate 1 kilogram at one meter per second per second—is equivalent to 3.6 ounces, or about the weight of a small apple.

[3] The electrostatic repulsion or attraction between two charged particles is another example of a central force.

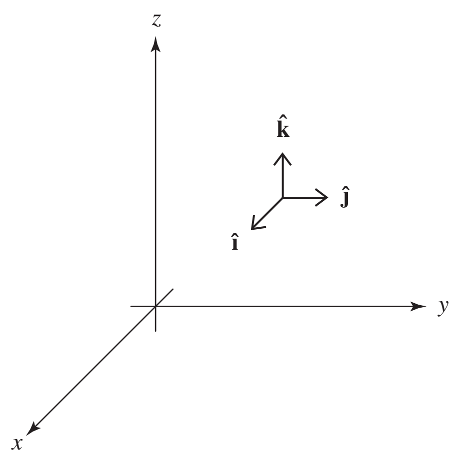


FIGURE 3.1 Axes and unit vectors in a Cartesian coordinate system.

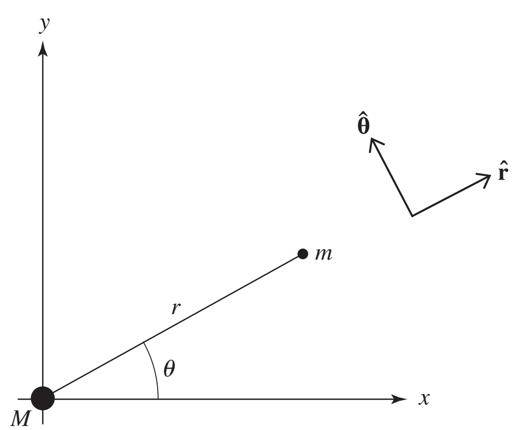


FIGURE 3.2 Axes and unit vectors in a polar coordinate system.

Figure 3.2, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \cos\theta + \hat{\mathbf{j}} \sin\theta \tag{3.2}$$

and

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta. \tag{3.3}$$

The dot product (or scalar product) of these unit vectors is

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0, \tag{3.4}$$

and their cross product (or vector product) is

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \hat{\mathbf{k}}(\cos^2\theta + \sin^2\theta) = \hat{\mathbf{k}}, \tag{3.5}$$

thus demonstrating that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are mutually orthogonal as well as being orthogonal to $\hat{\mathbf{k}}$, the unit vector in the z direction.

From equations (3.2) and (3.3), we see that

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \frac{d}{d\theta}(\hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta) = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta = \hat{\boldsymbol{\theta}} \tag{3.6}$$

and

$$\frac{d\hat{\boldsymbol{\theta}}}{d\theta} = \frac{d}{d\theta}(-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta) = -\hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta = -\hat{\mathbf{r}}. \tag{3.7}$$

We can then apply the chain rule to find the rate of change of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \hat{\boldsymbol{\theta}}\frac{d\theta}{dt} \tag{3.8}$$

and

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\hat{\mathbf{r}}\frac{d\theta}{dt}. \tag{3.9}$$

Note that since $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors, they change only in direction, not in magnitude.

The velocity of the planet can be expressed in polar coordinates as

$$\vec{\mathbf{v}} \equiv \frac{d\vec{\mathbf{r}}}{dt} = \frac{d(r\hat{\mathbf{r}})}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = v_r\hat{\mathbf{r}} + v_t\hat{\boldsymbol{\theta}}, \tag{3.10}$$

where

$$v_r = \frac{dr}{dt} \tag{3.11}$$

is the **radial velocity** and

$$v_t = r\frac{d\theta}{dt} \tag{3.12}$$

is the **tangential velocity**.

The angular momentum of the planet is defined as

$$\vec{\mathbf{L}} \equiv \vec{\mathbf{r}} \times \vec{\mathbf{p}}, \tag{3.13}$$

where $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ is the linear momentum. The rate of change of the angular momentum is then

$$\frac{d\vec{\mathbf{L}}}{dt} = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{v}} \times m\vec{\mathbf{v}} + \vec{\mathbf{r}} \times m\frac{d\vec{\mathbf{v}}}{dt}. \tag{3.14}$$

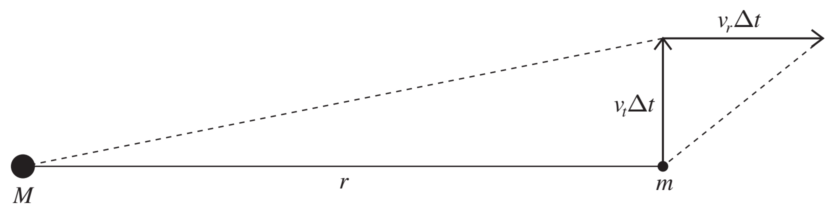


FIGURE 3.3 The motions of a planet during a short time interval Δt.

From Newton's second law of motion, we know that $md\vec{\mathbf{v}}/dt = \vec{\mathbf{F}}$. Thus, equation (3.14) can be rewritten as

$$\frac{d\vec{\mathbf{L}}}{dt} = m(\vec{\mathbf{v}} \times \vec{\mathbf{v}}) + \vec{\mathbf{r}} \times \vec{\mathbf{F}}. \tag{3.15}$$

However, $\vec{\mathbf{v}} \times \vec{\mathbf{v}} = 0$ (that's just a vector identity), and for a central force, $\vec{\mathbf{F}}$ is parallel to $\vec{\mathbf{r}}$ and thus $\vec{\mathbf{F}} \times \vec{\mathbf{r}} \propto \vec{\mathbf{r}} \times \vec{\mathbf{r}} = 0$. We conclude that for gravity or any other central force, angular momentum is conserved:

$$\frac{d\vec{\mathbf{L}}}{dt} = 0. \tag{3.16}$$

Note that the direction as well as the magnitude of $\vec{\mathbf{L}}$ is constant; this tells us that the motion of an object moving under the influence of a central force is confined to a plane.

The conservation of angular momentum is equivalent to Kepler's second law; to demonstrate that this is true, we use equation (3.10) to write the angular momentum explicitly as

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}} = mrv_t\hat{\mathbf{k}} = L\hat{\mathbf{k}}, \tag{3.17}$$

where v_t is the tangential velocity. Referring to Figure 3.3, consider a planet of mass m; at a time t, it is at a distance r from the Sun, which has mass M. During a brief time interval Δt, the planet moves a distance $v_t\Delta t$ in the tangential direction and a distance $v_r\Delta t$ in the radial direction. The area ΔA swept out by the planet–Sun line during this brief interval can be approximated as the sum of two triangles:

$$\Delta A \approx \frac{1}{2}r(v_t\Delta t) + \frac{1}{2}(v_r\Delta t)(v_t\Delta t), \tag{3.18}$$

where the two terms represent the left-hand triangle and the right-hand triangle in Figure 3.3.[4] In the limit $v_r\Delta t \ll r$, the right-hand triangle is vanishingly small compared to the left-hand triangle, and the area swept out can be further simplified as

$$\Delta A \approx \frac{1}{2}r(v_t\Delta t). \tag{3.19}$$

[4] In Figure 3.3, we are looking at the specific case $v_r > 0$, but performing a time reversal will yield the case $v_r < 0$.

The rate at which the planet–Sun line sweeps out area can then be written

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r v_t. \tag{3.20}$$

However, since we know that $L = mrv_t$, from equation (3.17), we can rewrite equation (3.20) in the form

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m}. \tag{3.21}$$

Since L and m are constant, so is the rate dA/dt at which the planet–Sun line sweeps out area. In other words, we have demonstrated that Kepler's second law will be true for a body acting under any central force, not just the force of gravity.

3.1.2 Kepler's First Law

To demonstrate that Kepler's first law follows from Newton's law of universal gravitation, we will have to demonstrate that the trajectory $r(\theta)$ of the mass m (the planet) constitutes an ellipse with the larger mass M (the Sun) at one focus. Using equations (3.12) and (3.17), we can write the angular momentum per unit mass of the orbiting body as

$$\frac{L}{m} = r^2 \frac{d\theta}{dt}, \tag{3.22}$$

which is constant for any central force. If the force acting on the mass m is gravitational, then from Newton's law of universal gravitation and second law of motion,

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} = m\frac{d\vec{\mathbf{v}}}{dt}. \tag{3.23}$$

The orbital acceleration under the influence of gravity is then

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{r^2}\hat{\mathbf{r}}. \tag{3.24}$$

From equation (3.9), we know that

$$\hat{\mathbf{r}} = -\left(\frac{d\theta}{dt}\right)^{-1} \frac{d\hat{\boldsymbol{\theta}}}{dt}. \tag{3.25}$$

By combining equations (3.24) and (3.25), we find that the acceleration of the planet is

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{GM}{r^2}\left(\frac{d\theta}{dt}\right)^{-1} \frac{d\hat{\boldsymbol{\theta}}}{dt}. \tag{3.26}$$

Combining this equation with equation (3.22), we see

$$\frac{L}{GMm}\frac{d\vec{\mathbf{v}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{dt}. \tag{3.27}$$

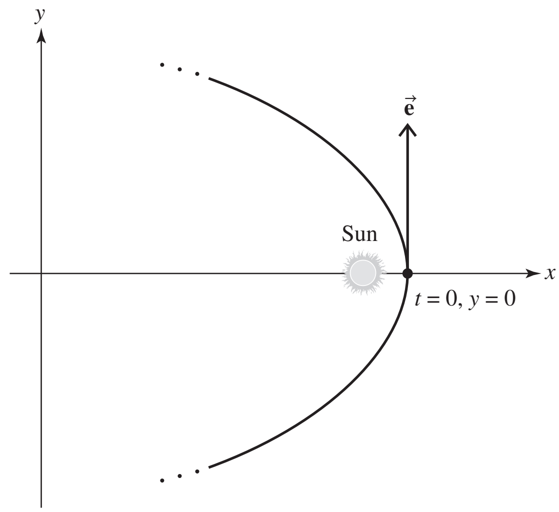


FIGURE 3.4 Time $t = 0$ corresponds to perihelion passage, with the planet crossing the x axis with its velocity in the positive y direction.

Integration of this simple differential equation yields

$$\frac{L}{GMm}\vec{\mathbf{v}} = \hat{\boldsymbol{\theta}} + \vec{\mathbf{e}}, \tag{3.28}$$

where $\vec{\mathbf{e}}$ is a constant of integration that depends on the initial conditions of the orbiting planet. We may choose the initial conditions for our own convenience. Let's choose the time $t = 0$ to correspond to a perihelion passage of the planet, and orient the axes so that perihelion passage occurs on the positive x axis (Figure 3.4). With this choice of coordinates, $\vec{\mathbf{v}}$ and $\hat{\boldsymbol{\theta}}$ both point in the y direction at $t = 0$; thus, we may write $\vec{\mathbf{e}} = e\hat{\mathbf{j}}$, where e is a constant. Equation (3.28) is then

$$\frac{L}{GMm}\vec{\mathbf{v}} = \hat{\boldsymbol{\theta}} + e\hat{\mathbf{j}}. \tag{3.29}$$

We now take the dot product of this equation and the unit vector $\hat{\boldsymbol{\theta}}$:

$$\frac{L}{GMm}\vec{\mathbf{v}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} + e\hat{\mathbf{j}} \cdot \hat{\boldsymbol{\theta}}. \tag{3.30}$$

To simplify the right-hand side of equation (3.30), we use equation (3.3) to find that $\hat{\mathbf{j}} \cdot \hat{\boldsymbol{\theta}} = \cos\theta$. To simplify the left-hand side, we write

$$\vec{\mathbf{v}} \cdot \hat{\boldsymbol{\theta}} = \left[v_r\hat{\mathbf{r}} + v_t\hat{\boldsymbol{\theta}}\right] \cdot \hat{\boldsymbol{\theta}} = v_t. \tag{3.31}$$

But, since equation (3.17) tells us that $mrv_t = L$, we may write

$$\vec{\mathbf{v}} \cdot \hat{\boldsymbol{\theta}} = v_t = \frac{L}{mr}. \tag{3.32}$$

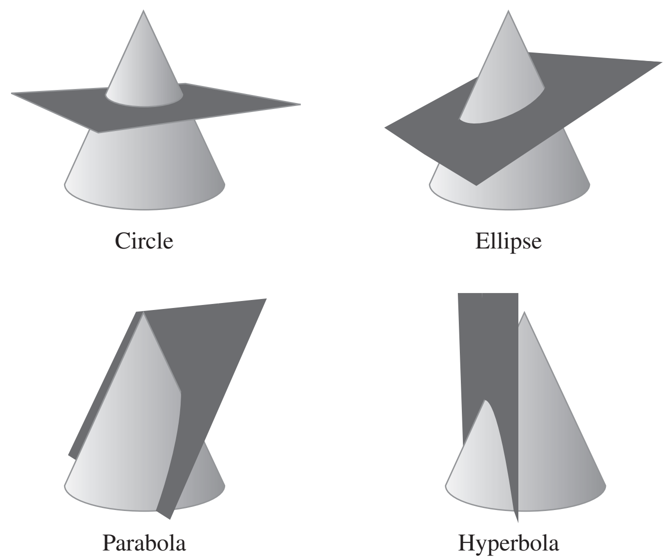


FIGURE 3.5 Conic sections demonstrated by slicing a cone.

Substituting equation (3.32) back into equation (3.30), we find a relationship between r and θ for fixed values of M, m, L, and e:

$$\frac{L^2}{GMm^2r} = 1 + e\cos\theta, \tag{3.33}$$

which can also be written in the form

$$r = \frac{L^2}{GMm^2(1 + e\cos\theta)}. \tag{3.34}$$

Equation (3.34) is the equation of a **conic section** in polar coordinates; as such, it provides a generalization of Kepler's first law.

Conic sections can be obtained by slicing a cone with a plane, as illustrated in Figure 3.5. If the plane is perpendicular to the cone's axis, then the conic section is a **circle**; from equation (3.34), we see that a circle corresponds to the special case $e = 0$, and hence $r = L^2/(GMm^2) =$ constant. If the slicing plane is tilted from the perpendicular by an angle less than the half-opening angle of the cone, the conic section obtained is an **ellipse**; this corresponds to the special case $0 < e < 1$.[5] When the slicing plane is tilted from the perpendicular by an angle exactly equal to the half-opening angle of the cone, the conic section resulting is a **parabola**; this is the special case $e = 1$. Finally, when the slicing plane is tilted by a larger angle, the conic section that results is a **hyperbola**,

[5] Yes, the parameter e in equation (3.34) is the same as the eccentricity e that we encountered while discussing elliptical orbits in Section 2.5, that is, the distance between foci divided by the major axis length.

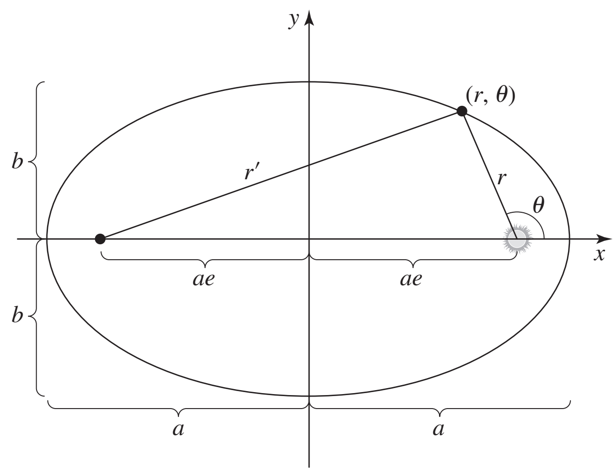


FIGURE 3.6 An ellipse of semimajor axis a and semiminor axis b.

which has $e > 1$. Kepler's first law is thus a special case that deals with **closed orbits**; that is, orbits with $e < 1$, which form closed curves (ellipses or circles). The basic physics of gravitation, however, permits **open orbits** as well, that is, parabolic or hyperbolic orbits with $e \geq 1$.

We have blithely asserted that the parameter e in equation (3.34), when it lies in the range $0 \leq e < 1$, is precisely the same as the eccentricity of an ellipse, defined as the distance between the foci divided by the length of the major axis. It is time to support that assertion by looking at the properties of ellipses in more depth. In Figure 3.6, an ellipse is shown along with a set of Cartesian coordinates; the origin of the coordinates is the center of the ellipse; the x axis lies along the major axis of the ellipse; and the y axis lies along the minor axis. We also define a system of polar coordinates centered on one of the foci. Let's call the focus at the origin the **principal focus** and require that it be the focus where the Sun is located, if the ellipse is regarded as a planetary orbit. The angular coordinate θ is measured counterclockwise from the x axis in the manner shown in Figure 3.6. The semimajor axis has length a and the semiminor axis has length b; each of the foci is displaced from the origin of the Cartesian coordinates by a distance ae. An arbitrary point on the ellipse is displaced by a distance r from the principal focus and a distance r' from the other focus; the basic property of an ellipse is that $r + r'$ is constant. By considering the two points of the ellipse lying on the x axis ($x = \pm a$, $y = 0$), we find that $r + r' = 2a$. It also follows that the perihelion distance, if the ellipse is regarded as a planetary orbit, is $q = a(1 - e)$ and the aphelion distance is $Q = a(1 + e)$.

Consider the point of the ellipse that lies on the positive y axis, where $r = r' = a$ as shown in Figure 3.7. From the Pythagorean theorem, as applied to the right triangle drawn in the figure, we find that $b^2 + (ae)^2 = r^2$, or since $r = a$,

$$b^2 = a^2(1 - e^2). \tag{3.35}$$

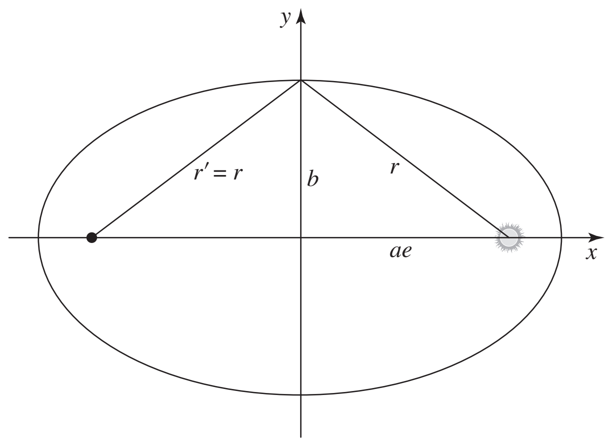


FIGURE 3.7 The relationship among the semimajor axis a, the semiminor axis b, and the eccentricity e.

This enables us to translate between the axis ratio of an ellipse, b/a, and its eccentricity,

$$e = (1 - b^2/a^2)^{1/2}. \tag{3.36}$$

It can also be shown that the average distance of all points on the ellipse from either focus is equal to the semimajor axis length a. To prove this, consider an arbitrary point on the ellipse, $P(x, y)$, and its reflection across the y axis, $P'(-x, y)$, as shown in Figure 3.8. The distance from point P to the focus on the positive x axis is r. By symmetry, the distance from the complementary point P' to the focus on the positive x axis is r', where r' is the distance from point P to the focus on the negative x axis. The average distance of the two points from the focus on the positive x axis is then

$$\langle r \rangle = \frac{r + r'}{2} = \frac{2a}{2} = a. \tag{3.37}$$

Since this relation holds for all (P, P') pairs, regardless of the choice of P, it is true that the average distance $\langle r \rangle$ from the focus over the entire ellipse is a.

Let us now describe the ellipse in terms of the polar coordinates (r, θ), where r is the distance from the principal focus and θ is the polar angle measured counterclockwise from the positive x axis, as shown in Figure 3.9. (When the ellipse represents an orbit, the angle θ is called the **true anomaly**.) Note in the figure that we can draw a triangle from the principal focus at $r = 0$, to an arbitrary point (r, θ) on the ellipse, to the other focus, then back to the principal focus. The internal angle of the vertex at the principal focus (as shown in Figure 2.17) is $\pi - \theta$. We can thus use the law of cosines to write

$$r'^2 = r^2 + (2ae)^2 - 2(2ae)r\cos(\pi - \theta). \tag{3.38}$$

Using the trigonometric identity $\cos(\pi - \theta) = -\cos\theta$, this becomes

$$r'^2 = r^2 + 4a^2e^2 + 4aer\cos\theta. \tag{3.39}$$

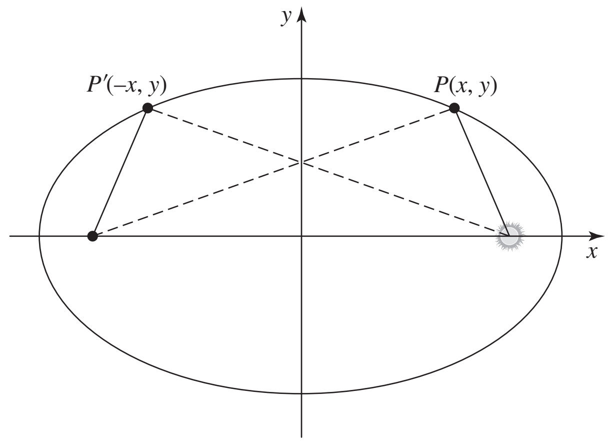


FIGURE 3.8 The point $P(x, y)$ is at a distance r from the focus on the positive x axis and a distance r' from the other focus. The complementary point $P'(-x, y)$ is at a distance r' from the focus on the positive x axis and a distance r from the other focus.

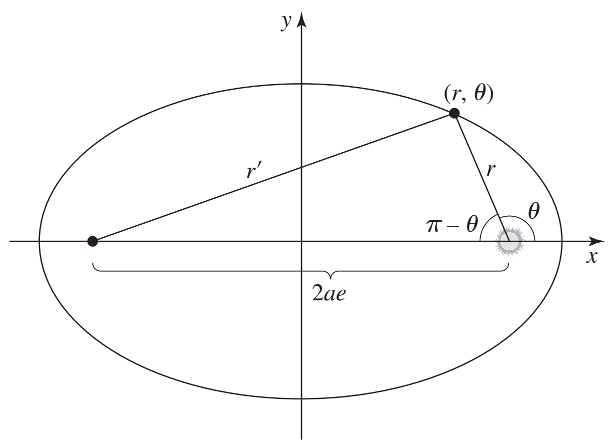


FIGURE 3.9 An ellipse in polar coordinates.

However, from the definition of the ellipse, we know that $r' = 2a - r$, which yields (squaring each side of the equation)

$$r'^2 = 4a^2 - 4ar + r^2. \tag{3.40}$$

Since the right-hand sides of equations (3.39) and (3.40) are equal, this tells us

$$4a^2e^2 + 4aer\cos\theta = 4a^2 - 4ar. \tag{3.41}$$

After dividing by $4a$ and doing a bit of rearranging, we find

$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \tag{3.42}$$

This equation for r as a function of θ is the equation for an ellipse in polar coordinates, with the origin at one focus. This is equivalent in form to equation (3.34), which gives the shape of an orbit if Newton's law of universal gravitation holds true. Comparison of equations (3.34) and (3.42) tells us that the angular momentum L of a planet's orbital motion is related to the size and shape of its orbit by the relation

$$\frac{L^2}{m^2} = GMa(1-e^2). \tag{3.43}$$

Since $L = mrv_t$, this relation can also be written in the form

$$r^2 v_t^2 = GMa(1-e^2). \tag{3.44}$$

When a planet is at perihelion, its velocity is entirely tangential ($v_{pe} = v_t$), and its distance from the Sun is $q = a(1-e)$. This implies that for a planet at perihelion,

$$v_{pe}^2 a^2 (1-e)^2 = GMa(1-e^2), \tag{3.45}$$

or

$$v_{pe} = \left[\frac{GM}{a}\frac{1+e}{1-e}\right]^{1/2}. \tag{3.46}$$

A similar analysis of the planet's speed at aphelion, where its velocity is also entirely tangential ($v_{ap} = v_t$), tells us that

$$v_{ap} = \left[\frac{GM}{a}\frac{1-e}{1+e}\right]^{1/2}. \tag{3.47}$$

3.1.3 Kepler's Third Law

Kepler's second law (equation 3.21) tells us that the area swept out per unit time by the planet–Sun line is a constant, $L/(2m)$. The area swept out in one orbital period, P, is the area of the ellipse, given by the standard formula $A = \pi ab$. For one complete orbital period, then, we may write

$$\frac{\pi ab}{P} = \frac{L}{2m}. \tag{3.48}$$

By squaring this equation and making the substitution $b^2 = a^2(1-e^2)$, we have

$$\frac{\pi^2 a^4 (1-e^2)}{P^2} = \frac{L^2}{4m^2}. \tag{3.49}$$

Since equation (3.43) gives us a relation among L, a, and e, namely,

$$\frac{L^2}{m^2} = GMa(1 - e^2), \tag{3.50}$$

we can substitute back into equation (3.49) to find

$$\frac{\pi^2 a^4 (1 - e^2)}{P^2} = \frac{GMa(1 - e^2)}{4}, \tag{3.51}$$

or

$$P^2 = \frac{4\pi^2}{GM} a^3, \tag{3.52}$$

which we recognize as Kepler's third law, $P^2 = Ka^3$, with the proportionality constant $K \propto 1/M$. With somewhat more exertion, taking into account the acceleration of the Sun (mass M) as well as the lower-mass planet (mass m), it is possible to reach the more general form

$$P^2 = \frac{4\pi^2}{G(M + m)} a^3. \tag{3.53}$$

Within the solar system, however, even the most massive of the planets, Jupiter, has a mass only 1/1000 that of the Sun, so the approximation $M + m \approx M$ is adequate.

The masses of celestial bodies are measured by how they accelerate nearby masses. In particular, we can use the orbital periods and semimajor axes of the planets to determine the mass of the Sun:

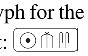
$$M = \frac{4\pi^2 a^3}{GP^2}. \tag{3.54}$$

The orbital period of the Earth, for instance, is 365.256 days $\times$ 86,400 s day$^{-1} = 3.16 \times 10^7$ s.[6] The semimajor axis of the Earth's orbit is $a = 1\ \text{AU} = 1.496 \times 10^{11}$ m. Thus, we can compute the mass of the Sun as

$$\begin{aligned} M &= \frac{4\pi^2 (1.496 \times 10^{11}\ \text{m})^3}{6.67 \times 10^{-11}\ \text{m}^3\ \text{s}^{-2}\ \text{kg}^{-1} (3.16 \times 10^7\ \text{s})^2} \\ &= 1.98 \times 10^{30}\ \text{kg} \equiv 1 M_\odot. \end{aligned} \tag{3.55}$$

Later in this book, we will find that the solar mass ($M_\odot$) is a useful unit for expressing the masses of stars (and larger objects).[7]

[6] A useful approximation is that the length of the year is $\pi \times 10^7$ s.

[7] The "dot in a circle" symbol $\odot$ is the standard astronomical symbol for the Sun. It is of great antiquity, being identical to the Egyptian hieroglyph for the Sun god Ra, seen here, for instance, as the first syllable in the name of the pharaoh Ramses the Great: 

3.2 ▪ ORBITAL ENERGETICS

Suppose you place a particle of mass m at a location $\vec{\mathbf{r}}$ relative to an object of mass M; you give it a kick so that it is initially moving at a velocity $\vec{\mathbf{v}}$. What determines whether its orbit is closed (a circle or ellipse, with $e < 1$) or open (a parabola or hyperbola, with $e \geq 1$)? In a sense, it's all about the energy. The particle will have an energy E that is the sum of its kinetic energy K and its gravitational potential energy U:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{3.56}$$

The square of the velocity can be determined by squaring equation (3.28):

$$\left(\frac{L}{GMm}\right)^2 \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} + 2e\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{j}} + e^2\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}$$

$$\left(\frac{L}{GMm}\right)^2 v^2 = 1 + 2e\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{j}} + e^2. \tag{3.57}$$

Since $\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{j}} = \cos\theta$, from equation (3.3), we may now write the kinetic energy as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{GMm}{L}\right)^2 (1 + e^2 + 2e\cos\theta). \tag{3.58}$$

The kinetic energy is greatest at perihelion ($\theta = 0$), which is as it should be, since that's when the particle is moving fastest. Now using equation (3.34) for r as a function of θ, we can write the potential energy as

$$U = -\frac{GMm}{r} = -\frac{(GM)^2m^3}{L^2}(1 + e\cos\theta). \tag{3.59}$$

The amplitude of the potential energy, $|U|$, is greatest at perihelion ($\theta = 0$), which is as it should be, since that's when the particle is closest to the mass M. By adding together the kinetic energy (equation 3.58) and the potential energy (equation 3.59), and doing a bit of rearranging, we find

$$E = \left(\frac{GMm}{L}\right)^2 \frac{m}{2}(e^2 - 1). \tag{3.60}$$

This is constant, which is as it should be, since energy is conserved for this isolated two-body system. We can also, if we so choose, write the orbital eccentricity as a function of energy E and angular momentum L:

$$e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}. \tag{3.61}$$

We can readily identify three distinct cases:

1. **Hyperbolic orbits:** As we recall from our discussion of conic sections (page 68), the case $e > 1$ represents a hyperbola. Equation (3.60) shows that $e > 1$ corresponds

to a total energy $E > 0$; that is, $K > |U|$. This is an open orbit; the mass m is not gravitationally bound to the mass M. The mass m makes a single perihelion passage at $\theta = 0$ and does not return—its value of r, the distance from the mass M, continues to increase monotonically after perihelion passage.

2. **Parabolic orbits:** In the case where $e = 1$ exactly, the mass m is marginally unbound to M; that is, its velocity approaches zero asymptotically as r approaches infinity. In the case of a parabolic orbit, equation (3.60) shows that $e = 1$ corresponds to $E = 0$, or $K = |U|$. Equation (3.56) reveals that a particle will be on a parabolic orbit if its speed is equal to the **escape speed**:

$$v_{\rm esc}(r) = \left(\frac{2GM}{r}\right)^{1/2}. \tag{3.62}$$

If its velocity is greater than $v_{\rm esc}$, it will be on a hyperbolic orbit.

3. **Elliptical orbits:** In the case where $e < 1$, the mass m is gravitationally bound; it goes around the mass M on an elliptical orbit. The total energy, when $e < 1$, is $E < 0$, corresponding to $K < |U|$. The special case $e = 0$ corresponds to a perfectly circular orbit. Equation (3.60) shows that a circular orbit is the orbit that minimizes the energy E for a given angular momentum L.

3.3 ▪ ORBITAL SPEED

It is not possible in general to obtain a simple equation that gives the time dependence of a planet's distance from the Sun, $r(t)$, or orbital speed, $v(t)$.[8] However, it is possible to find the orbital speed v as a simple function of r, which can be useful. We start with the equation for a conic section (equation 3.42), which we write in the form

$$e \cos\theta = \frac{a(1-e^2) - r}{r}. \tag{3.63}$$

The orbital speed as a function of θ is given by equation (3.58):

$$v^2 = \frac{2K}{m} = \left(\frac{GMm}{L}\right)^2 (1 + e^2 + 2e\cos\theta). \tag{3.64}$$

Thus, by combining equations (3.63) and (3.64), we find an equation that gives the orbital speed as a function of r:

$$v^2 = \frac{G^2M^2m^2}{L^2}\left(1 + e^2 + \frac{2}{r}[a(1-e^2) - r]\right). \tag{3.65}$$

Using equation (3.43), which tells us $L^2/m^2 = GMa(1-e^2)$, we find

[8] This also implies that there is no simple equation for $\theta(t)$, since if we had one, we could use the conic section equation for $r(\theta)$ to find $r(t)$.

$$v^2 = \frac{G^2M^2}{GMa(1-e^2)}\left(\frac{r + e^2r + 2a(1-e^2) - 2r}{r}\right)$$
$$= \frac{GM}{a(1-e^2)}\left(\frac{2a(1-e^2) - r(1-e^2)}{r}\right)$$
$$= \frac{GM}{a}\left(\frac{2a}{r} - 1\right) = GM\left(\frac{2}{r} - \frac{1}{a}\right). \quad (3.66)$$

The resulting equation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \quad (3.67)$$

is called the **vis viva** equation. The Latin term *vis viva*, which translates literally to "living force," is an archaic bit of scientific terminology that was first employed by Gottfried Leibniz (best known as the other discoverer of calculus). Leibniz used the term *vis viva* to refer to the quantity mv^2, what we would now call $2K$, or twice the kinetic energy. The *vis viva* equation is a statement of how the kinetic energy of an orbiting object changes as a function of r. By using Kepler's third law (equation 3.52), we can also write the *vis viva* equation in the form

$$v(r) = \frac{2\pi a}{P}\left(2\frac{a}{r} - 1\right)^{1/2}. \quad (3.68)$$

This implies that the orbital angular speed $\omega = v/r$ of a planet is

$$\omega(r) = \frac{2\pi}{P}\frac{a}{r}\left(2\frac{a}{r} - 1\right)^{1/2}. \quad (3.69)$$

At perihelion, where $r = q = a(1-e)$, the angular speed of the planet is

$$\omega_{\rm pe} = \frac{2\pi}{P}\frac{(1+e)^{1/2}}{(1-e)^{3/2}}, \quad (3.70)$$

and at aphelion, where $r = Q = a(1+e)$, the angular speed is

$$\omega_{\rm ap} = \frac{2\pi}{P}\frac{(1-e)^{1/2}}{(1+e)^{3/2}}. \quad (3.71)$$

Here on Earth, for instance, the observed average angular speed of the Sun along the ecliptic is equal to 2π radians per sidereal year, or $\omega = 0.986°/\text{day}$. However, since the Earth's orbit has an eccentricity $e = 0.017$, the observed angular speed is greatest at the time of perihelion (early January), when $\omega_{\rm pe} = 1.020°/\text{day}$, and smallest at the time of aphelion (early July), when $\omega_{\rm ap} = 0.953°/\text{day}$.

An interesting application of the *vis viva* equation (eq. 3.68) addresses the problem of the **transfer orbit**. In traveling from the Earth to another planet, the transfer orbit is the route you would take from the Earth to the other planet's orbit. The **Hohmann transfer orbit**, illustrated in Figure 3.10, is an ellipse whose perihelion is at the orbit of

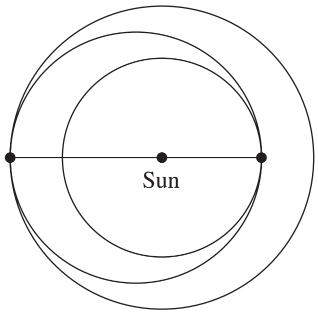


FIGURE 3.10 A Hohmann transfer orbit for interplanetary travel (here from Earth to Mars). The transfer orbit is an ellipse with its perihelion at Earth and its aphelion at the orbit of Mars.

the inner planet and whose aphelion is at the orbit of the outer planet. As the German engineer Walter Hohmann pointed out in the 1920s, the Hohmann transfer orbit has two desirable properties. First, it requires only two engine burns when done properly: one when leaving Earth and one when the destination planet is reached. The rest of the time, the spacecraft is "coasting" on a Newtonian orbit. Second, it is economical in its fuel use; launching your spacecraft on a hyperbolic orbit will cause it to reach its destination faster but requires more energy.

As a concrete example, suppose you want to send a spacecraft to Mars. As a first approximation, we can assume that the orbit of the Earth is a circle of radius $a_\oplus = 1\ \text{AU} = 1.50 \times 10^8$ km, with orbital period $P_\oplus = 1\ \text{yr} = 3.16 \times 10^7$ s.[9] We further assume that the orbit of Mars is a larger circle, of radius $a_{\text{Mars}} = 1.52 a_\oplus = 2.27 \times 10^8$ km, with orbital period $P_{\text{Mars}} = 1.88\ \text{yr} = 5.94 \times 10^7$ s. The semimajor axis of the Hohmann transfer orbit from Earth to Mars is

$$a_{\text{to}} = \frac{a_\oplus + a_{\text{Mars}}}{2} = \frac{1\ \text{AU} + 1.52\ \text{AU}}{2} = 1.26\ \text{AU}. \tag{3.72}$$

The orbital period for the transfer orbit is then

$$P_{\text{to}}[\text{ yr}] = (a[\text{ AU}])^{3/2} = (1.26)^{3/2} = 1.41. \tag{3.73}$$

Traveling from Earth to Mars requires half an orbit, or a time $t = P_{\text{to}}/2 = 0.71\ \text{yr} \approx$ 260 days.

The average speed of the Earth on its orbit is

$$v_\oplus = \frac{2\pi a_\oplus}{P_\oplus} = \frac{2\pi(1.50 \times 10^8\ \text{km})}{3.16 \times 10^7\ \text{s}} = 29.8\ \text{km s}^{-1}. \tag{3.74}$$

[9] The "cross in a circle" symbol $\oplus$ is the standard astronomical symbol for the Earth.

The average speed of Mars is slower:

$$v_{\rm Mars} = \frac{2\pi a_{\rm Mars}}{P_{\rm Mars}} = \frac{2\pi(2.27 \times 10^8 \text{ km})}{5.94 \times 10^7 \text{ s}} = 24.0 \text{ km s}^{-1}. \tag{3.75}$$

When the spacecraft has just left the Earth, it is at the perihelion of the Hohmann transfer orbit. Its speed, from the *vis viva* equation (eq. 3.68), is

$$\begin{aligned} v_{\rm pe} &= \frac{2\pi a_{\rm to}}{P_{\rm to}} \left(\frac{2a_{\rm to}}{a_\oplus} - 1 \right)^{1/2} \\ &= \frac{2\pi(1.26 \text{ AU})(1.50 \times 10^8 \text{ km AU}^{-1})}{(1.41 \text{ yr})(3.16 \times 10^7 \text{ s yr}^{-1})} \left[\frac{2(1.26 \text{ AU})}{1.00 \text{ AU}} - 1 \right]^{1/2} \\ &= 26.7 \text{ km s}^{-1}(1.52)^{1/2} = 32.9 \text{ km s}^{-1}. \end{aligned} \tag{3.76}$$

Thus, at the perihelion of the Hohmann transfer orbit, the spacecraft must be going *faster* than the Earth by an amount $\Delta v = v_{\rm pe} - v_\oplus = 3.1 \text{ km s}^{-1}$. When the spacecraft is just reaching Mars, it is at the aphelion of the Hohmann transfer orbit. Its speed, from equation (3.68), is then

$$v_{\rm ap} = \frac{2\pi a_{\rm to}}{P_{\rm to}} \left(\frac{2a_{\rm to}}{a_{\rm Mars}} - 1 \right)^{1/2} = 26.7 \text{ km s}^{-1} \left[\frac{2(1.26 \text{ AU})}{1.52 \text{ AU}} - 1 \right]^{1/2} = 21.7 \text{ km s}^{-1}. \tag{3.77}$$

Thus, in order to match its velocity to that of Mars, the spacecraft must increase its speed by $\Delta v = v_{\rm Mars} - v_{\rm ap} = 2.3 \text{ km s}^{-1}$. (If you want your spacecraft to go into orbit around Mars, like the *Mars Reconnaissance Orbiter,* the time, direction, and duration of your engine burn depend on the orbital parameters you want to attain.)

Use of a Hohmann transfer orbit requires careful timing. If you are sending a spacecraft to Mars, for instance, the craft must reach the aphelion of its orbit just as Mars reaches that point. This restricts launches to certain times, known as **launch windows**. If you fail to launch during one launch window, you could wait for one synodic period of the target planet before launching again. For a mission to Mars, whose synodic period is 2.1 years, this could be a frustrating wait.

3.4 ▪ THE VIRIAL THEOREM

If a system contains only two spherical bodies, such as a star and planet, there is a simple analytic solution (first seen in Section 2.5) for the planet's trajectory, $r(\theta)$. Similarly, Section 3.2 yields simple formulas for the planet's kinetic energy $K(\theta)$ and potential energy $U(\theta)$, while Section 3.3 gives the *vis viva* equation for v as a function of r. In a system containing more than two bodies, however, there are no longer any simple analytic solutions for the bodies' properties. Thus, when astronomers study large stellar systems such as star clusters and galaxies, they generally use numerical techniques to compute the stellar orbits using a computer. However, despite the complexity of many-body systems such as star clusters, it is possible to find useful statistical results that describe the average global properties of the system. One such result is the **virial theorem**, which relates the total kinetic energy of a system to its total potential energy.

To derive the virial theorem, let's suppose we have a system containing N stars (or planets, or other compact massive bodies). The mass of the ith star is m_i, and its location is $\vec{\mathbf{r}}_i$. We can define a function

$$A \equiv \sum_{i=1}^{N} m_i \frac{d\vec{\mathbf{r}}_i}{dt} \cdot \vec{\mathbf{r}}_i. \tag{3.78}$$

The reason for defining this function starts to become a bit more obvious when we take the time derivative of A:

$$\frac{dA}{dt} = \sum_{i=1}^{N} \left(m_i \frac{d\vec{\mathbf{r}}_i}{dt} \cdot \frac{d\vec{\mathbf{r}}_i}{dt} + m_i \frac{d^2\vec{\mathbf{r}}_i}{dt^2} \cdot \vec{\mathbf{r}}_i \right). \tag{3.79}$$

The first term on the right-hand side of equation (3.79) is twice the kinetic energy, and the second term can be transformed using Newton's second law,

$$m_i \frac{d^2\vec{\mathbf{r}}_i}{dt^2} = \vec{\mathbf{F}}_i, \tag{3.80}$$

where $\vec{\mathbf{F}}_i$ is the net force acting on the ith star. Thus, we may write

$$\frac{dA}{dt} = 2K + \sum_{i=1}^{N} \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i, \tag{3.81}$$

where K is the sum of the kinetic energies of all the stars in the system. The term $\sum \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i$ was named the **virial** by the physicist Rudolf Clausius.[10]

Equation (3.81) is the most general form of the virial theorem. It applies to any system of bodies that follow Newton's second law, regardless of the forces $\vec{\mathbf{F}}_i$ acting on them. A more useful form of the virial theorem can be found by taking the time average of equation (3.81). If we average over the time interval $t = 0 \to t = \tau$, we find

$$\begin{aligned} 2\langle K \rangle + \langle \sum_{i=1}^{N} \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i \rangle &= \langle \frac{dA}{dt} \rangle \\ &= \frac{1}{\tau} \int_0^{\tau} \frac{dA}{dt} dt \\ &= \frac{A(\tau) - A(0)}{\tau}. \end{aligned} \tag{3.82}$$

If the system is bound, then the velocity of each particle, as well as its displacement from the origin, remains finite. In that case, $A(t)$, as given by equation (3.78), is finite at all

[10] Clausius also coined the term "entropy," probably his most memorable contribution to the scientific vocabulary.

times, and the right-hand side of equation (3.82) goes to zero in the limit $\tau \to \infty$. Thus, for any bound system of particles, the time-averaged virial theorem has the form

$$2\langle K\rangle + \langle\sum_{i=1}^{N} \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i\rangle = 0. \tag{3.83}$$

The virial theorem as expressed in equation (3.83) can be applied to any bound system, for instance, to a gas of molecules enclosed within a box. However, as astronomers, we are interested in the specific case of an isolated bound stellar system, in which the force acting on the ith star is the sum of the gravitational forces exerted by the other $N-1$ stars in the system:

$$\vec{\mathbf{F}}_i = \sum_{j\neq i} \frac{Gm_i m_j(\vec{\mathbf{r}}_j - \vec{\mathbf{r}}_i)}{|\vec{\mathbf{r}}_j - \vec{\mathbf{r}}_i|^3}. \tag{3.84}$$

For such a system, what is the value of the virial, $\sum \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i$? Let's start with a simple system containing only two stars. For this system, the virial will be

$$\begin{aligned}\vec{\mathbf{F}}_1 \cdot \vec{\mathbf{r}}_1 + \vec{\mathbf{F}}_2 \cdot \vec{\mathbf{r}}_2 &= \frac{Gm_1m_2(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)\cdot\vec{\mathbf{r}}_1}{|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|^3} + \frac{Gm_2m_1(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)\cdot\vec{\mathbf{r}}_2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|^3} \\ &= -\frac{Gm_1m_2|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|^2}{|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|^3} \\ &= -\frac{Gm_1m_2}{|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|}.\end{aligned} \tag{3.85}$$

The right-hand side of equation (3.85) is simply the potential energy U of the two-star system. By extension, for a three-star system, the virial will be equal to the sum of the potential energies of all three pairs: (1,2), (2,3), and (3,1). For a system containing N stars, the virial will be equal to the sum of the potential energies of all $N_{\text{pair}} = N(N-1)/2$ pairs of stars that can be drawn from the system. We can thus write

$$\sum_{i=1}^{N} \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i = U = \sum_{i=1}^{N}\sum_{j>i} -\frac{Gm_im_j}{|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|}, \tag{3.86}$$

and the virial equation (eq. 3.83) becomes

$$2\langle K\rangle + \langle U\rangle = 0. \tag{3.87}$$

The virial theorem is useful to astronomers, as we find in Section 20.2, when it enables us to estimate the mass of distant galaxies.

PROBLEMS

3.1 Comet Hale-Bopp has an orbit about the Sun with eccentricity $e = 0.9951$ and semimajor axis length $a = 186.5$ AU. What is the sidereal orbital period of Comet Hale-Bopp? What is Comet Hale-Bopp's distance from the Sun at perihelion? What is its distance from the Sun at aphelion? Comet Hale-Bopp passed through perihelion on 1997 April 1; did the previous perihelion passage of Comet Hale-Bopp occur before or after the birth of Aristotle?

3.2 The asteroid Eros is seen in opposition from the Earth once every 847 days. What is the sidereal orbital period of Eros? What is the length a of the semimajor axis of Eros' orbit? The eccentricity of the orbit of Eros is $e = 0.223$. Does Eros ever come within 1 AU of the Sun?

3.3 Consider a satellite in a circular, low-Earth orbit; that is, the satellite's elevation above the Earth's surface is $h \ll R_\oplus$. Show that the orbital period P for such a satellite is approximately

$$P = C\left(1 + \frac{3h}{2R_\oplus}\right).$$

What is the numerical value of the constant C in minutes? When Puck, in *A Midsummer Night's Dream*, boasted, "I'll put a girdle round about the Earth in forty minutes" (Act 2, Scene 1), could he have done so by traveling on a circular orbit, accelerated by the Earth's gravity alone? If so, what would be his elevation h?

3.4 What is the orbital period for a low-*lunar* orbit (as was used by the Apollo command modules)?

3.5 (a) Io is the innermost Galilean satellite of Jupiter. The orbital period of Io is $P = 1.769$ days; the semimajor axis of its orbit is $a = 421{,}600$ km (slightly larger than the Moon's orbit about the Earth). Given this information, find the mass of Jupiter.
(b) Phobos is the inner moon of Mars. The orbital period of Phobos is $P = 0.32$ days; the semimajor axis of its orbit is $a = 9370$ km. Find the mass of Mars. (Hint: you may assume the masses of Io and Phobos are negligible compared to those of their parent planets.)

3.6 Communications and weather satellites are often placed in *geosynchronous* orbits. A geosynchronous orbit is an orbit about the Earth with orbital period P exactly equal to one sidereal day. What is the semimajor axis $a_{\rm gs}$ of a geosynchronous orbit? What is the orbital velocity $v_{\rm gs}$ of a satellite on a circular geosynchronous orbit?

3.7 Starting with the equation for an ellipse in polar coordinates (eq. 3.42), derive the more familiar Cartesian form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

3.8 The *Hubble Space Telescope (HST)* is on a circular, low-Earth orbit, at an elevation $h = 600$ km above the Earth's surface. What is its orbital period? For an observer who sees *HST* pass through the zenith, how long is *HST* above the horizon during each orbit?

3.9 One way of lifting a satellite into geosynchronous orbit is to use the space shuttle to lift it into a circular, low-Earth orbit (with $h = 300$ km above the Earth's surface), and then use a booster rocket to place the satellite on a Hohmann transfer orbit (see Section 3.3) up to a circular geosynchronous orbit. What is the orbital velocity v_{ss} of the satellite while it is still in low-Earth orbit? What is the orbital velocity at pericenter, v_{pe}, of the appropriate Hohmann transfer orbit? What is the orbital velocity at apocenter, v_{ap}, of the Hohmann transfer orbit? How long does it take the satellite to travel from the low-Earth orbit to the geosynchronous orbit?

3.10 A small particle of mass m is on a circular orbit of radius R around a much larger mass M. Suppose that we suddenly increase the speed at which the mass m is moving, by a factor α (that is, $v_{\rm final} = \alpha v_{\rm initial}$, with $\alpha > 1$). Compute the major axis, minor axis, pericenter distance, and apocenter distance for the new orbit; express your answers in terms of R and α alone.